The sum $\Sigma_{i=0}^{b} \frac{b!}{(b-i)!i!}$

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Abstract

In this, we explore a couple of proofs of the sum $\sum_{i=0}^{b} \frac{b!}{(b-i)!i!}$.

1 Background

My friend Matt had a homework question about finding a bound $\Theta(g(n))$ for the function $(n+a)^b$ for $n > n_0$. Choosing the convenient $n_0 = a$, and intuiting $g(n) = n^b$, we can easily demonstrate the existence of a constant c_2 for which $c_2 n^b > (n+a)^b$ for $n > n_0 = a$:

$$(n+a)^b = n^b + \frac{b!}{(b-1)!1!} n^{(b-1)} a + ... + \frac{b!}{1!(b-1)!} n a^{(b-1)} + a^b \tag{1}$$

At $n = n_0 = a$, this is constant is determined:

$$(a+a)^b = a^b + \frac{b!}{(b-1)!1!} a^b + ... + \frac{b!}{1!(b-1)!} a^b + a^b = a^b \sum_{i=0}^{b} \frac{b!}{(b-i)!i!}$$

This isn't exactly a pretty sum to deal with, and so I sought to find a simpler equivalent expression:

$$(a+a)^b = (2a)^b = 2^b a^b = a^b \sum_{(i=0)}^{b} \frac{b!}{(b-i)!i!}$$

However, I wanted to prove this using induction.

2 Proof by Induction

First, we demonstrate that this hold true for a base case. We'll use $b = 1, 2, 3, 4$:

$$\begin{array}{rcccl} b=1 & : & 1+1 & = & 2^1 \\ b=2 & : & 1+2+1 & = & 2^2 \\ b=3 & : & 1+3+3+1 & = & 2^3 \\ b=4 & : & 1+4+6+4+1 & = & 2^4 \end{array}$$

Next, we'll assume that this is true for $b = k$, and then using that, demonstrate that the relationship holds for $b = k + 1$.

$$\sum_{i=0}^{k} \frac{k!}{(k-i)!i!} = 2^k$$

We must show that this implies that:

$$\sum_{i=0}^{k+1} \frac{(k+1)!}{(k+1-i)!i!} = 2^{k+1} \tag{2}$$

Let

$$f(i,k) = \frac{k!}{(k-i)!i!}$$

and note that

$$f(i,k) = f(i-1,k-1) + f(i,k-1)$$

for $1 \leq i \leq k-1$ and $k \geq 1$. You can note this by looking at the above base cases and recognizing that each element is the sum of the two elements above it. For example, in the $b = 4$ case, the 6 is the sum of the 3 and 3 from the $b = 3$ case. For a more detailed demonstration that this is true, please see appendix A.

For $i = 0$ and $i = k$ where $k \geq 1$, note that $f(i,k) = 1$. Thus, $f(0,k+1) = f(0,k)$ and $f(k+1,k+1) = f(k,k)$. As such,

$$\sum_{i=0}^{k+1} f(i,k+1) = f(0,k+1) + f(1,k+1) + ... + f(k,k+1) + f(k+1,k+1)$$

$$= [f(0,k)] + [f(0,k) + f(1,k)] + ... + [f(k-1,k) + f(k,k)] + [f(k,k)]$$

$$= 2f(0,k) + 2f(1,k) + ... + 2f(k-1,k) + 2f(k,k)$$

$$= 2\sum_{i=0}^{k} f(i,k)$$

Plugging this back into equation 2, we get:

$$\sum_{i=0}^{k+1} f(i,k+1) = 2\sum_{i=0}^{k} f(i,k) = 2 \times 2^k = 2^{k+1}$$

A $f(i,k) = f(i-1,k-1) + f(i,k-1)$

For $1 \leq i \leq k-1$ and $k \geq 1$,

$$\frac{k!}{(k-i)!i!} = \frac{(k-1)!}{(k-i)!(i-1)!} + \frac{(k-1)!}{(k-1-i)!i!}$$

Multiplying by each term in the denominator ($i!$, $(k-i)!$, $(i-1)!$ and $(k-1-i)!$),

$$k!(i-1)!(k-1-i)! = (k-1)!i!(k-1-i)! + (k-1)!(i-1)!(k-i)!$$

Dividing by $(k-1)!$,

$$k(i-1)!(k-1-i)! = i!(k-1-i)! + (i-1)!(k-i)!$$

Dividing by $(i-1)!$,

$$k(k-1-i)! = i(k-1-i)! + (k-i)!$$

Dividing by $(k-1-i)!$,

$$k = i + (k-i) = k$$